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On Titchmarsh–Weyl $M(\lambda)$-Functions for Linear Hamiltonian Systems

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1. INTRODUCTION

In his expository article [20], Weyl states that the idea which led him to the limit-point or limit-circle classification of second-order equations [19] was the construction of the Green's function with $\operatorname{Im}\lambda \neq 0$ for the singular boundary value problem

$$-(py')' + qy = \lambda ry + f, \qquad 0 \leqslant x < \infty. \tag{1.1}$$

It is assumed a boundary condition at 0 is imposed on (1.1); a central question arises then as to whether (1.1) is well-posed. Weyl restricts (1.1) to a compact interval $[0, b]$, and then imposes a boundary condition at b. This leads to the construction of a circle C_b whose points correspond one-one with the set of all possible self-adjoint boundary conditions at b. The circles C_b are nested as $b \uparrow \infty$, and the resolution of whether a boundary condition at infinity is needed to make (1.1) well-posed is answered by determining if the circles converge to a circle (a boundary condition at infinity is then needed) or converge to a point (no boundary condition is needed).

Classical Titchmarsh–Weyl theory is concerned with the existence of integrable square solutions of (1.1) and with the nature of the spectra of boundary value problems associated with (1.1). Beginning with special solutions $\theta(0, \lambda)$ and $\phi(x, \lambda)$ of (1.1) with initial values

$$\theta(0, \lambda) = \cos\alpha, \qquad p(0)\,\theta'(0, \lambda) = \sin\alpha,$$

$$\phi(0, \lambda) = \sin\alpha, \qquad p(0)\,\phi'(0, \lambda) = -\cos\alpha.$$

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Titchmarsh shows in [17] (where references to his earlier work are given) that there always exists a function $m(\lambda)$, analytic at least in the upper and lower half-planes, such that the solution

$$\psi(x, \lambda) = \theta(x, \lambda) + m(\lambda)\, \phi(x, \lambda), \qquad 0 < x < \infty,$$

is of integrable square, i.e., $\psi(\cdot, \lambda) \in \mathscr{L}^2[0, \infty)$. The function $m(\lambda)$ has become known as the Titchmarsh–Weyl coefficient [10, p. 50]. To each well-posed boundary value problem associated with (1.1) there corresponds a unique $m(\lambda)$ whose singularities constitute the spectrum of the problem. For example, eigenvalues are simple poles of $m(\lambda)$. To a non-well-posed problem there corresponds a family of $m(\lambda)$ functions which, for fixed λ, comprises the Weyl limit circle.

For an historical account and survey of results of the m-coefficient for second-order symmetric differential expressions, we refer to the recent article of Everitt and Bennewitz [10].

The investigation of the associated eigenfunction expansions leads to certain difficulties because of the possible presence of a continuous spectrum. Many authors have worked on these eigenfunction expansions and related spectral questions since the publication of Weyl's original work [19]. In particular, the text by Titchmarsh [17] contains a detailed account of the second-order equation. Numerous contributions were made by Hartman and Wintner and their school and by the Russian school. An extensive exposition on the evolution of the spectral theory of ordinary differential equations may be found in [3, pp. 1581–1628].

More recently, Everitt (cf. [4–6]) has carried much of the classic Titchmarsh–Weyl theory over to higher-order formally symmetric scalar differential equations. In particular, we refer the reader to the survey papers [7, 8] by Everitt and Kumar. In this work a central role is played by the analytic function $m(\lambda)$ which give rise to the square-integrable solutions of the differential equation.

We consider here a systems formulation of a singular boundary value problem. The formulation is that used by Atkinson [1, Chap. 9] for self-adjoint systems. It includes the linear Hamiltonian system

$$\begin{aligned} y' &= Ay + Bz, \\ z' &= Cy - A^*z + \lambda K y, \end{aligned} \tag{1.2}$$

where $B^* = B$, $C^* = C$, and $K^* = K$. Atkinson's formulation also includes the general symmetric operator of order $2n$

$$M[y]=\frac{1}{w}\left\{P_n y+\sum_{k=1}^{n}\left[(-)^k (P_{n-k}y^{(k)})^{(k)}\right.\right.$$
$$\left.\left.-i(q_{n-k}y^{(k)})^{(k-1}-i(q_{n-k}y^{(k-1)})^{(k)}\right]\right\}$$

as well as the corresponding odd-order symmetric operator. This was proved by Walker in [18. The most general scalar symmetric differential expressions appear to be the quasi-differential expressions of Shin and Zettl [9]. As noted in [9, pp. 392–399], these general expressions may also be put in the systems formulations at Atkinson.

Atkinson's system is considered in this paper under a hypothesis which for scalar equations is the limit-point or minimal deficiency index case. Large classes of scalar equations have the minimal deficiency index property. Under this hypothesis we show in Section 3 that there is a unique matrix function $M(\lambda)$ which corresponds to the scalar function $m(\lambda)$ above. In Section 4 we use the results of Section 3 to study the singular boundary value problem for Atkinson's system and construct the associated Green's matrix.

A corollary of Section 3 is that we are able to extend the notion of a principal solution, cf. [2, 12, 15], to the system (1.2) when the parameter λ is nonreal. This principal solution with complex λ behaves in much the same manner as the principal solution of (1.2) for $\lambda=0$ and the system nonoscillatory.

2. Preliminaries

We quote from [1, Chap. 9] most of the definitions and basic facts we require concerning linear Hamiltonian systems. The system we study has the form

$$Jy'=[\lambda A(x)+B(x)]\,y, \qquad a\leqslant x<b^*, \quad b^*\leqslant\infty, \tag{2.1}$$

where J, $A(x)$, $B(x)$ are complex matrices, λ is a complex parameter, and $y(x)$ is a $k\times 1$ vector function. To put (1.2) in the form (2.1), see [2, p. 34]. Sometimes, $y(x)$ will be replaced by a $k\times r$ solution matrix $Y(x)$. Following [1], we take $A(x)$ and $B(x)$ to be locally integrable over $[a, b^*)$, and J a constant nonsingular matrix, so that the usual existence and uniqueness theorems hold for (2.1) when appropriate initial values are assigned. Additionally, J will be skew-Hermitian, while $A(x)$ and $B(x)$ are Hermitian, that is,

$$J^*=-J, \qquad A^*(x)=A(x), \qquad B^*(x)=B(x), \tag{2.2}$$

where $*$ denotes the complex conjugate transpose.Moreover, $A(x)$ is assumed to be nonnegative-definite, written

$$A(x) \geqslant 0 \tag{2.3}$$

by which is meant $u^*A(x)u \geqslant 0$ for all complex $k \times 1$ vectors u. Lastly, we assume the "definiteness" condition

$$\int_\alpha^\beta y^*(x)\,A(x)\,y(x)\,dx > 0, \qquad a \leqslant \alpha < \beta < b^*, \tag{2.4}$$

for each solution $y(x)$ of (2.1) which does not vanish identically.

Recall the identity [1, p. 253]

$$y^*(b)\,Jy(b) - y^*(a)\,Jy(a) = (\lambda - \bar{\lambda}) \int_a^b y^*(x)\,A(x)\,y(x)\,dx, \qquad b < b^*, \tag{2.5}$$

valid for solutions y of (2.1). When $y \not\equiv 0$ and $\lambda \neq \bar{\lambda}$, (2.4) implies that the right side of (2.5) is nonzero.

On an interval $[a, b]$, $a < b < b^*$, admissible boundary value problems associated with (2.1) are given by matrices M and N satisfying the self-adjointness condition

$$M^*JM = N^*JN, \tag{2.6}$$

together with the condition that $Mv = Nv = 0$ must imply $v = 0$, where v is a column vector. The boundary value problem consists of finding a solution y of (2.1) such that

$$y(a) = Mv, \qquad y(b) = Nv \tag{2.7}$$

for some vector $v \neq 0$. For example, the problem

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix}' = \left\{ \lambda \begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} q & 0 \\ 0 & r^{-1} \end{bmatrix} \right\} \begin{pmatrix} u \\ v \end{pmatrix},$$

$$\begin{pmatrix} u(a) \\ v(a) \end{pmatrix} = \begin{bmatrix} 0 & \sin\alpha \\ 0 & \cos\alpha \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \qquad \begin{pmatrix} u(b) \\ v(b) \end{pmatrix} = \begin{bmatrix} \sin\beta & 0 \\ \cos\beta & 0 \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

is equivalent to the classical Sturm–Liouville problem

$$(ry')' + (\lambda p + q)\,y = 0,$$
$$y(a)\cos\alpha - r(a)\,y'(a)\sin\alpha = 0 = y(b)\cos\beta - r(b)\,y'(b)\sin\beta$$

when u, v, y are related by $u = y$, $v = ry'$. The role of the vector v is to "parameterize" the boundary values of y.

A solution y of (2.1) is said to be of "integrable square" if

$$\int_a^{b^*} y^*(x) A(x) y(x)\, dx < \infty. \tag{2.8}$$

The set of all such solutions forms a complex vector space, and so there arises the question of the number of linearly independent solutions of integrable square. In this connection we cite the following result [1, p. 295].

THEOREM A. *Let J/i have k' negative eigenvalues and k'' positive eigenvalues ($k' + k'' = k$). Then* (2.1) *has at least k' linearly independent solutions satisfying* (2.8) *if* $\operatorname{Im}\lambda > 0$ *and at least k'' such solutions if* $\operatorname{Im}\lambda < 0$.

To determine exactly the number of linearly independent solutions of integrable square, we need additional hypotheses. In this paper, J will subsequently have one of two forms. If k is even, $k = 2m$, then

$$J = J_{\text{even}} = \begin{bmatrix} 0 & -J_1^* \\ J_1 & 0 \end{bmatrix}, \tag{2.9}$$

where J_1 is an $m \times m$ nonsingular matrix. If k is odd, $k = 2m + 1$, then we take

$$J = J_{\text{odd}} = \begin{bmatrix} 0 & 0 & -J_1^* \\ 0 & i & 0 \\ J_1 & 0 & 0 \end{bmatrix}, \tag{2.10}$$

for the same J_1. The zero symbol in (2.9) and (2.10) stands for the $m \times m$ zero matrix, except for the middle row and middle column of (2.10) where it denotes the $1 \times m$ and $m \times 1$ zero vector, respectively. It can be calculated that J/i has m negative eigenvalues and m positive eigenvalues in case (2.9), and m negative eigenvalues and $(m + 1)$ positive eigenvalues if (2.10) holds. Finally we will assume the "limit-point" condition ($\operatorname{Im}\lambda \neq 0$)

$$y^* J z = 0 \tag{2.11}$$

for all y and z of integrable square, where y solves (2.1) and z satisfies

$$Jz' = [\bar{\lambda} A(x) + B(x)]z, \qquad a \leqslant x < b^*. \tag{2.12}$$

For such y and z, y^*Jz is in any event constant, as may be verified by direct differentiation. In the case where (2.1) represents an even-order scalar differential equation, say

$$Lu = \sum_{k=0}^{n} (-1)^k [P_k u^{(k)}]^{(k)} = 0$$

(see [2, p. 76] for the precise formulation) $y^*(x) Jz(x)$ is the usual bilinear form $[u\, v](x)$ associated with Green's formula, where u and v correspond to y and z in the formulation. Condition (2.11) is then equivalent to the operator L being of limit-point at b^* [14, p. 19].

In regard to the number of linearly independent integrable square solutions, we make the following definitions. For $\operatorname{Im} \lambda \neq 0$, let $S(\lambda)$ be the set of integrable square solutions of (2.1). Then $S(\bar{\lambda})$ is the set of integrable square solutions of the "conjugate" equation (2.12). By Theorem A and our hypotheses on the eigenvalues of J/i, we know that $\dim S(\lambda) \geqslant m$ and $\dim S(\bar{\lambda}) \geqslant m$ if $k = 2m$ is even, but if $k = 2m + 1$ is odd, then $\dim S(\lambda) \geqslant m$ and $\dim S(\bar{\lambda}) \geqslant m + 1$ for $\operatorname{Im} \lambda > 0$.

LEMMA 2.1. *Under the limit point condition* (2.11) *we have*

$$\dim S(\lambda) + \dim S(\bar{\lambda}) = k, \qquad \operatorname{Im} \lambda \neq 0. \tag{2.13}$$

Proof. By previous remarks, $\dim S(\lambda) + \dim S(\bar{\lambda}) \geqslant k$. Assume $\dim S(\lambda) + \dim S(\bar{\lambda}) > k$ for some λ. Since J is nonsingular, then $\dim JS(\bar{\lambda}) = \dim S(\bar{\lambda})$. For fixed x, each of $JS(\bar{\lambda})$ and $S(\lambda)$ lies in $\mathbf{C}^k$, the k-dimensional complex vector space of $k \times 1$ complex vectors. By the assumption, there must exist a $y \in S(\lambda)$ which is not orthogonal to $JS(\bar{\lambda})$, that is, $y^*(x) Jz(x) \neq 0$ for every $z \in S(\bar{\lambda})$. This contradicts (2.11) and proves the lemma.

COROLLARY 2.1. *If* $\operatorname{Im} \lambda \neq 0$, *then*

$$\dim S(\lambda) = \dim S(\bar{\lambda}) = m \qquad \text{if} \quad k = 2m;$$

$$\dim S(\lambda) = m, \qquad \dim S(\bar{\lambda}) = m + 1 \qquad \text{if} \quad k = 2m + 1, \quad \operatorname{Im} \lambda > 0.$$

Note that (2.1) cannot be "regular" at b^*, as otherwise k solutions would be of integrable square. Following [1], we let $Y(x, \lambda)$ denote the "fundamental matrix" solution to (2.1), that is, Y is a $k \times k$ matrix whose columns solve (2.1),

$$JY' = [\lambda A(x) + B(x)] Y, \qquad a \leqslant x < b^*, \quad Y(a, \lambda) = I, \tag{2.14}$$

where I is the $k \times k$ identity. We will partition Y according to the partity of k and the sign of $\operatorname{Im} \lambda$ as follows. We will always write

$$Y = \begin{bmatrix} \Theta & \Phi \\ \hat{\Theta} & \hat{\Phi} \end{bmatrix}, \qquad \Theta = \Theta(x, \lambda), \qquad \Phi = \Phi(x, \lambda), \qquad \text{etc.,} \tag{2.15}$$

but require that

$$\Theta \text{ and } \Phi \text{ are } m \times m \text{ if } k = 2m \text{ and } \operatorname{Im} \lambda \neq 0; \tag{2.16}$$

$$\Theta \text{ is } (m+1) \times m \text{ and } \Phi \text{ is } (m+1) \times (m+1) \text{ if } k = 2m+1 \text{ and } \operatorname{Im} \lambda > 0; \tag{2.17}$$

$$\Theta \text{ is } m \times (m+1) \text{ and } \Phi \text{ is } m \times m \text{ if } k = 2m+1 \text{ and } \operatorname{Im} \lambda < 0. \tag{2.18}$$

In all cases, $\hat{\Theta}$ and $\hat{\Phi}$ are to be given the same number of columns as Θ and Φ, respectively.

LEMMA 2.2. *The matrix $\Phi(x, \lambda)$ is in all cases nonsingular for $x > a$ and* $\operatorname{Im} \lambda \neq 0$.

Proof. We will proceed by cases, basing arguments on the matrix version of (2.5),

$$W^*(b) JW(b) - W^*(a) JW(a) = (\lambda - \bar{\lambda}) \int_a^b W^*(x) A(x) W(x)\, dx, \tag{2.19}$$

where $W(x)$ is a $k \times r$ matrix whose columns are solutions to (2.1).

Case (i): $k = 2m$. Since J is given by (2.9) and (2.16) holds, we have

$$\begin{bmatrix} \Phi \\ \hat{\Phi} \end{bmatrix}^* J \begin{bmatrix} \Phi \\ \hat{\Phi} \end{bmatrix} = [\Phi^*, \hat{\Phi}^*] \begin{bmatrix} 0 & -J_1^* \\ J_1 & 0 \end{bmatrix} \begin{bmatrix} \Phi \\ \hat{\Phi} \end{bmatrix} = \hat{\Phi}^* J_1 \Phi - \Phi^* J_1 \hat{\Phi}.$$

By the initial conditions in (2.14) and the partitioning (2.15) in the present case, $\Phi(a) = \Phi^*(a) = 0$. Thus (2.19) becomes

$$(1/i)(\hat{\Phi}^* J_1 \Phi - \Phi^* J_1 \hat{\Phi})(b) = 2 \operatorname{Im} \lambda \int_a^b [\Phi^*, \hat{\Phi}^*]\, A [\Phi^*, \hat{\Phi}^*]^*\, dx. \tag{2.20}$$

From (2.4) we see that the right side of (2.20) is nonzero, its sign being that of $\operatorname{Im} \lambda$, with inequality in the sense of (2.3). There could not exist an $m \times 1$ vector u such that $\Phi(b) u = 0$, else

$$u^*(\hat{\Phi}^* J_1 \Phi - \Phi^* J_1 \hat{\Phi})(b) u = u^* \hat{\Phi}^*(b) J_1 (\Phi(b) u) - (\Phi(b) u)^* J_1 \hat{\Phi}(b) u = 0.$$

Therefore $\Phi(b)$ is nonsingular for $b > a$.

Case (ii): $k = 2m + 1$, $\operatorname{Im} \lambda > 0$. This time (2.10) and (2.17) hold and we accordingly represent Φ in the form

$$\begin{bmatrix} \Phi \\ \\ \hat{\Phi} \end{bmatrix} = \begin{bmatrix} W_1 \\ W_2 \\ \hat{\Phi} \end{bmatrix},$$

where W_1 is $m \times (m+1)$ and W_2 is $1 \times (m+1)$. Then

$$\begin{bmatrix} \Phi \\ \hat{\Phi} \end{bmatrix}^* J \begin{bmatrix} \Phi \\ \hat{\Phi} \end{bmatrix} = [W_1^*, W_2^*, \hat{\Phi}^*] \begin{bmatrix} 0 & 0 & -J_1^* \\ 0 & i & 0 \\ J_1 & 0 & 0 \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \\ \hat{\Phi} \end{bmatrix}$$

$$= \hat{\Phi}^* J_1 W_1 + i W_2^* W_2 - W_1^* J_1 \hat{\Phi}.$$

The initial values of Φ given $W_1(a) = 0$ (the $m \times (m+1)$ matrix) and $W_2(a) = (1, 0, \ldots, 0)$. Therefore (2.19) takes the form

$$(1/i)(\hat{\Phi}^* J_1 W_1 + i W_2^* W_2 - W_1^* J_1 \hat{\Phi})(b)$$

$$= W_2^* W_2(a) + 2 \operatorname{Im} \lambda \int_a^b [\Phi^*, \hat{\Phi}^*] A [\Phi^*, \hat{\Phi}^*]^* \, dx.$$

Again, the right-hand side is positive, in the sense of (2.3). If $\Phi(b)u = 0$ for some $(m+1) \times 1$ vector u, then $W_1(b)u = 0$ and $W_2(b)u = 0$ also. Arguing as before, we obtain

$$u^*(\hat{\Phi}^* J_1 W_1 + i W_2^* W_2 - W_1^* J_1 \Phi)(b)u = 0,$$

which is a contradiction. Therefore $\Phi(b)$ must be nonsingular.

Case (iii): $k = 2m+1$, $\operatorname{Im} \lambda < 0$. Here we partition $\hat{\Phi}$, which is $(m+1) \times m$, as

$$\begin{bmatrix} \Phi \\ \\ \hat{\Phi} \end{bmatrix} = \begin{bmatrix} \Phi \\ W_3 \\ W_4 \end{bmatrix},$$

where W_3 is $1 \times m$ and W_4 is $m \times m$. Then

$$\begin{bmatrix} \Phi \\ \hat{\Phi} \end{bmatrix}^* J \begin{bmatrix} \Phi \\ \hat{\Phi} \end{bmatrix} = [\Phi^*, W_3^*, W_4^*] \begin{bmatrix} 0 & 0 & -J_1^* \\ 0 & i & 0 \\ J_1 & & 0 \end{bmatrix} \begin{bmatrix} \Phi \\ W_3 \\ W_4 \end{bmatrix}$$

$$= W_4^* J_1 \Phi + i W_3^* W_3 - \Phi^* J_1 W_4.$$

The initial values of this matrix are $i W_3^*(a) W_3(a) = 0$. Thus

$$(1/i)(W_4^* J_1 \Phi + i W_3^* W_3 - \Phi^* J_1 W_4)(b)$$

$$= 2 \operatorname{Im}(\lambda) \int_a^b [\Phi^*, \hat{\Phi}^*] A [\Phi^*, \hat{\Phi}^*]^* \, dx.$$

Now $\Phi(b)u = 0$, for an $m \times 1$ vector u, yields $u^* W_3^* W_3 u \geqslant 0$ for the left-hand side. However, the right side is negative due to the sign of $\operatorname{Im}(\lambda)$. This completes the proof of the lemma.

Note that the above proof applies also to the matrices $\hat{\Phi}(x, \lambda)$, $\Theta(x, \lambda)$ and $\hat{\Theta}(x, \lambda)$ in the even-order case. So for $k = 2m$ we conclude that $\hat{\Phi}$, Θ and $\hat{\Theta}$ are invertible for $\operatorname{Im}(\lambda) \neq 0$.

In analogy to the classical Titchmarsh–Weyl coefficient [4, 5] we define the functions $M_b(\lambda)$ by

$$M_b(\lambda) = -\Phi^{-1}(b, \lambda)\, \Theta(b, \lambda), \qquad b > a, \quad \operatorname{Im}(\lambda) \neq 0.$$

We are going to prove that as $b \to b^*$, $M_b(\lambda)$ converges to a limit function $M_\infty(\lambda)$ (Section 3) which is analytic in the upper and lower half-planes. It can be verified that $(M_\infty)_{rs}$ agrees with the corresponding m-coefficient m_{sr} of [5] when (2.1) represents an even-order symmetric differential expression with smooth coefficients. In the odd-order case considered in [6], M_∞ reduces to the m-coefficients p_{ij} and n_{ij} of Everitt to the extent that $(M_\infty)_{rs} = p_{sr}$ for $\operatorname{Im}\lambda > 0$ and $(M_\infty)_{rs} = n_{sr}$ for $\operatorname{Im}\lambda < 0$. Similar relations exists with the quantities m_{rs}, p_{rs}, and n_{rs} of [7, 8].

For $b > a$ and $\operatorname{Im}(\lambda) \neq 0$, define the solution $X_b(x, \lambda)$ of (2.1) by

$$X_b(x, \lambda) = \Theta(x, \lambda) + \Phi(x, \lambda)\, M_b(\lambda).$$

Then X_b has size $m \times m$ if $k = 2m$, and if $k = 2m + 1$, X_b is either $(m+1) \times m$ or $m \times (m+1)$ depending on whether $\operatorname{Im}(\lambda) > 0$ or $\operatorname{Im}(\lambda) < 0$. By the definition of $M_b(\lambda)$ we have

$$X_b(b, \lambda) = 0, \qquad b > a, \quad \operatorname{Im}\lambda \neq 0. \tag{2.21}$$

Noting (2.15) and the sizes of the matrices involved, define

$$\hat{X}_b(x, \lambda) = \hat{\Theta}(x, \lambda) + \hat{\Phi}(x, \lambda)\, M_b(\lambda).$$

LEMMA 2.3. *For $b > a$ and* $\operatorname{Im}\lambda \neq 0$, *we have*

$$\left(\frac{1}{i}\right)\begin{bmatrix} X_b(b) \\ \hat{X}_b(b) \end{bmatrix}^* J \begin{bmatrix} X_b(b) \\ \hat{X}_b(b) \end{bmatrix} \begin{cases} = 0 & \text{if } k = 2m \text{ or if } k = 2m+1 \text{ and } \operatorname{Im}\lambda > 0; \\ \geqslant 0 & \text{if } k = 2m+1 \text{ and } \operatorname{Im}\lambda < 0. \end{cases}$$

Proof. The proof proceeds by cases.

Case (i): $k = 2m$. As J is given by (2.9) we have by (2.21)

$$\begin{bmatrix} X_b(b) \\ \hat{X}_b(b) \end{bmatrix}^* J \begin{bmatrix} X_b(b) \\ \hat{X}_b(b) \end{bmatrix} = [0, \hat{X}_b(b)^*] \begin{bmatrix} 0 & -J_1^* \\ J_1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \hat{X}_b(b) \end{bmatrix} = 0.$$

Case (ii): $k = 2m+1, \operatorname{Im}\lambda > 0$. Noting that J is given by (2.10) we write

$$\begin{bmatrix} X_b \\ \\ \hat{X}_b \end{bmatrix} = \begin{bmatrix} Z_1 \\ Z_2 \\ \hat{X}_b \end{bmatrix},$$

where Z_1 is $m \times m$ and Z_2 is $1 \times m$. Then $Z_1(b) = 0$ and $Z_2(b) = 0$, and so

$$\begin{bmatrix} X_b(b) \\ \hat{X}_b(b) \end{bmatrix}^* J \begin{bmatrix} X_b(b) \\ \hat{X}_b(b) \end{bmatrix} = [0, 0, \hat{X}_b^*(b)] \begin{bmatrix} 0 & 0 & -J_1^* \\ 0 & i & 0 \\ J_1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \hat{X}_b(b) \end{bmatrix} = 0.$$

Case (iii): $k = 2m+1$, $\operatorname{Im}\lambda < 0$. This time we write

$$\begin{bmatrix} X_b \\ \hat{X}_b \end{bmatrix} = \begin{bmatrix} X_b \\ Z_3 \\ Z_4 \end{bmatrix},$$

where Z_3 is $1 \times (m+1)$ and Z_4 is $m \times (m+1)$. Then

$$\begin{bmatrix} X_b(b) \\ \hat{X}_b(b) \end{bmatrix}^* J \begin{bmatrix} X_b(b) \\ \hat{X}_b(b) \end{bmatrix} = [0, Z_3^*, Z_4^*] \begin{bmatrix} 0 & 0 & -J_1^* \\ 0 & i & 0 \\ J_1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ Z_3 \\ Z_4 \end{bmatrix} = iZ_3^* Z_3,$$

and so division by i results in a nonnegative matrix. This completes the proof.

The following lemma will be used to construct integrable square solutions of (2.1).

LEMMA 2.4. *Suppose that* $\{C_n\}$ *is a convergent sequence of matrices and* $\{b_n\}$ *is a sequence with* $b_n \to b^*$ *as* $n \to \infty$ *such that if*

$$\begin{pmatrix} X_n \\ \hat{X}_n \end{pmatrix} = \begin{pmatrix} \Theta & \Phi \\ \hat{\Theta} & \hat{\Phi} \end{pmatrix} \begin{pmatrix} I \\ C_n \end{pmatrix}, \tag{2.22}$$

then

$$\left(\frac{1}{i}\right) \begin{bmatrix} X_n(b_n) \\ \hat{X}_n(b_n) \end{bmatrix}^* J \begin{bmatrix} X_n(b_n) \\ \hat{X}_n(b_n) \end{bmatrix} = \begin{cases} 0 & \text{if } \operatorname{Im}\lambda > 0. \\ \geqslant 0 & \text{if } \operatorname{Im}\lambda < 0. \end{cases} \tag{2.23}$$

Then the sequence given by (2.22) *converges uniformly on compact intervals to a solution* $[X^*, \hat{X}^*]^*$ *of* (2.1) *of integrable square. Moreover,*

$$0 < \int_a^{b^*} \begin{bmatrix} X \\ \hat{X} \end{bmatrix}^* A \begin{bmatrix} X \\ \hat{X} \end{bmatrix} \leqslant \frac{i}{2 \operatorname{Im}\lambda} \begin{bmatrix} X(a) \\ \hat{X}(a) \end{bmatrix}^* J \begin{bmatrix} X(a) \\ \hat{X}(a) \end{bmatrix}.$$

Proof. The convergence of $\{C_n\}$ ensures that the sequence given by (2.22) converges uniformly on compact intervals to a solution $[X^*, \hat{X}^*]^*$ of (2.1). By (2.3), (2.5), and (2.23),

$$\begin{aligned} 0 &< \int_a^{b_n} \begin{bmatrix} X_n \\ \hat{X}_n \end{bmatrix}^* A \begin{bmatrix} X_n \\ \hat{X}_n \end{bmatrix} \\ &= \frac{1}{2i \operatorname{Im} \lambda} \left\{ \begin{bmatrix} X_n(b_n) \\ \hat{X}_n(b_n) \end{bmatrix}^* J \begin{bmatrix} X_n(b_n) \\ \hat{X}_n(b_n) \end{bmatrix} - \begin{bmatrix} X_n(a) \\ \hat{X}_n(a) \end{bmatrix}^* J \begin{bmatrix} X_n(a) \\ \hat{X}_n(a) \end{bmatrix} \right\} \\ &\leqslant \frac{i}{2 \operatorname{Im} \lambda} \begin{bmatrix} X_n(a) \\ \hat{X}_n(a) \end{bmatrix}^* J \begin{bmatrix} X_n(a) \\ \hat{X}_n(a) \end{bmatrix}. \end{aligned}$$

Thus if $b_n > b$

$$\begin{aligned} 0 &< \int_a^{b} \begin{bmatrix} X_n \\ \hat{X}_n \end{bmatrix}^* A \begin{bmatrix} X_n \\ \hat{X}_n \end{bmatrix} \leqslant \int_a^{b_n} \begin{bmatrix} X_n \\ \hat{X}_n \end{bmatrix}^* A \begin{bmatrix} X_n \\ \hat{X}_n \end{bmatrix} \\ &\leqslant \frac{i}{2 \operatorname{Im} \lambda} \begin{bmatrix} X_n(a) \\ \hat{X}_n(a) \end{bmatrix}^* J \begin{bmatrix} X_n(a) \\ \hat{X}_n(a) \end{bmatrix}. \end{aligned} \tag{2.24}$$

Letting $n \to \infty$ and then $b \to b^*$ in the left-most and right-most terms of (2.24) completes the proof.

3. Main Results

Associated with the boundary value problem (2.1)–(2.7) is the characteristic function [1, p. 268]

$$F_{MN} = F_{MN}(\lambda, b) = \tfrac{1}{2}[M + Y^{-1}(b)N][M - Y^{-1}(b)N]^{-1} J^{-1}. \tag{3.1}$$

The characteristic function is defined and analytic [1, p. 257] for λ not an eigenvalue of (2.1)–(2.7) and arises in the construction of the resolvent kernel [1, p. 268]. The basic properties of F_{MN} are summarized in the following [1, pp. 269, 289].

Theorem B. (i) *For each $b > a$ there is a compact set $C(b, \lambda)$ of complex $k \times k$ matrices such that $C(b_2, \lambda) \subset C(b_1, \lambda)$ for $b_2 > b_1$ and $F_{MN}(\lambda, b) \subset C(b, \lambda)$ for all M and N satisfying* (2.6). (ii) F_{MN} *satisfies*

$$\frac{1}{i} [F_{MN} - F_{MN}^*] \lesseqgtr 0 \qquad \text{for} \quad \operatorname{Im} \lambda \gtreqless 0. \tag{3.2}$$

The proof of part (i) also shows that if K is a compact set in either $\{\lambda : \operatorname{Im} \lambda > 0\}$ or $\{\lambda : \operatorname{Im} \lambda < 0\}$, then the compact set $C(b, \lambda)$ may be chosen independent of λ for λ in K.

A principal role in the study of F_{MN} is played by the matrix $\Phi^{-1}\Theta$. To utilize this dependence (cf. (3.4 and 3.5) below) we consider F_{MN} for certain special cases of M and N. We use I_m to denote the $m \times m$ identity matrix. The cases considered are:

(I) $$k = 2m, \qquad M = \begin{pmatrix} 0 & A_2^* \\ 0 & -A_1^* \end{pmatrix}, \qquad N = \begin{pmatrix} 0 & 0 \\ -I_m & 0 \end{pmatrix},$$

where A_1 and A_2 are $m \times m$ matrices satisfying $\operatorname{rank}[A_1, A_2] = m$ and $A_1 J_1 A_2^* = A_2 J_1^* A_1^*$;

(II) $$k = 2m, M \text{ as in (I)}, \qquad N = \begin{pmatrix} -I_m & 0 \\ 0 & 0 \end{pmatrix};$$

(III) $$k = 2m+1, \qquad M = \begin{pmatrix} 0 & 0 & A_2^* \\ 0 & 1 & 0 \\ 0 & 0 & -A_1^* \end{pmatrix}, \qquad N = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ -I_m & 0 & 0 \end{pmatrix},$$

where A_1 and A_2 are as in (I).

In cases (I) and (II) we have $M^*JM = N^*JN = 0$, while in case (III),

$$M^*JM = N^*JN = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

hence the boundary conditions are self-adjoint. Calculations also show that $Mv = Nv = 0$ implies $v = 0$ in all cases. We note that in (I) the boundary conditions $y(a) = Mv$, $y(b) = Nv$ are equivalent to

$$[A_1 J_1, A_2 J_1^*]\, y(a) = 0, \qquad [I_m, 0]\, y(b) = 0,$$

with similar equations holding in cases (II) and (III).

To calculate F_{MN} in these cases we use [1, p. 268]

$$F_{MN} J + \tfrac{1}{2} I = M(Y(b)M - N)^{-1}\, Y(b). \tag{3.3}$$

In case (I) the right-hand side of (3.3) is (all functions are evaluated at b)

$$\begin{aligned} &\begin{pmatrix} 0 & A_2^* \\ 0 & -A_1^* \end{pmatrix} \begin{pmatrix} 0 & \Theta A_2^* - \Phi A_1^* \\ I_m & \hat{\Theta} A_2^* - \hat{\Phi} A_1^* \end{pmatrix}^{-1} Y \\ &\quad = \begin{pmatrix} 0 & A_2^* \\ 0 & -A_1^* \end{pmatrix} \begin{pmatrix} --- & I_m \\ [\Theta A_2^* - \Phi A_1^*]^{-1} & 0 \end{pmatrix} \begin{pmatrix} \Theta & \Phi \\ \hat{\Theta} & \hat{\Phi} \end{pmatrix} \qquad (3.4) \\ &\quad = \begin{pmatrix} A_2^* W_1 & A_2^* W_2 \\ -A_1^* W_1 & -A_1^* W_2 \end{pmatrix} \end{aligned}$$

where

$$W_1 = [\Theta A_2^* - \Phi A_1^*]^{-1}\,\Theta = [A_2^* - \Theta^{-1}\Phi A_1^*]^{-1},$$
$$W_2 = [\Theta A_2^* - \Phi A_1^*]^{-1}\,\Phi = [\Phi^{-1}\Theta A_2^* - A_1^*]^{-1}.$$

In case (II) a similar calculation yields that the right-hand side of (3.3) is

$$\begin{pmatrix} A_2^*[A_2^* - \hat{\Theta}^{-1}\hat{\Phi}A_1^*]^{-1} & A_2^*[\hat{\Phi}^{-1}\hat{\Theta}A_2^* - A_1^*]^{-1} \\ -A_1^*[A_2^* - \hat{\Theta}^{-1}\hat{\Phi}A_1^*]^{-1} & -A_1^*[\hat{\Phi}^{-1}\hat{\Theta}A_2^* - A_1^*]^{-1} \end{pmatrix}. \tag{3.5}$$

We consider case (III) for $\operatorname{Im}\lambda > 0$. If we define the $m \times (m+1)$ and $(m+1)$ matrices $\tilde{A}_2, \tilde{A}_1$ by

$$\tilde{A}_2 = (0, A_2^*), \qquad \tilde{A}_1 = \begin{bmatrix} 1 & 0 \\ 0 & -A_1^* \end{bmatrix},$$

then as in case (I),

$$M[YM - N]^{-1}\,Y = \begin{bmatrix} 0 & \tilde{A}_2 \\ 0 & \tilde{A}_1 \end{bmatrix} \begin{bmatrix} 0 & \Theta\tilde{A}_2 + \Phi\tilde{A}_1 + C \\ I_m & \hat{\Theta}\tilde{A}_2 + \hat{\Phi}\tilde{A}_1 \end{bmatrix}^{-1} Y,$$

where C is the $(m+1) \times (m+1)$ matrix with $C_{m+1,1} = 1$ and other entries 0. Continuing as in case (I) yields that the right-hand side of (3.3) is

$$\begin{pmatrix} \tilde{A}_2[\Phi^{-1}\Theta\tilde{A}_2 + \tilde{A}_1 + \Phi^{-1}C]^{-1}\Phi^{-1}\Theta & \tilde{A}_2[\Phi^{-1}\Theta\tilde{A}_2 + \tilde{A}_1 + \Phi^{-1}C]^{-1} \\ \tilde{A}_1[\Phi^{-1}\Theta\tilde{A}_2 + \tilde{A}_1 + \Phi^{-1}C]^{-1}\,\Phi^{-1}\Theta & \tilde{A}_1[\Phi^{-1}\Theta\Phi\tilde{A}_2 + \tilde{A}_1 + \Phi^{-1}C]^{-1} \end{pmatrix} \tag{3.6}$$

LEMMA 3.1. *If K is a compact set in either $\{\lambda : \operatorname{Im}\lambda > 0\}$ or $\{\lambda : \operatorname{Im}\lambda < 0\}$ and $\varepsilon > 0$, then the matrix function $\Phi^{-1}(x,\lambda)\,\Theta(x,\lambda)$ is uniformly bounded for $a + \varepsilon \leqslant x < b^*$ and $\lambda \in K$. For $k = 2m$, $\hat{\Phi}^{-1}(x,\lambda)\,\hat{\Theta}(x,\lambda)$ is uniformly bounded for $a \leqslant x < b^*$ and $\lambda \in K$.*

Proof. For $k = 2m$, we take $A_2 = 0$, $A_1 = I_m$ in (3.4). Then

$$F_{MN}J + \tfrac{1}{2}I = \begin{pmatrix} 0 & 0 \\ \Phi^{-1}\Theta & I_m \end{pmatrix}$$

and the result is an immediate corollary of Theorem B and the remark following it. For $k = 2m$, $A_2 = 0$, $A_1 = I_m$ in (3.5) yields that

$$F_{MN}J + \tfrac{1}{2}I = \begin{pmatrix} 0 & 0 \\ \hat{\Phi}^{-1}\hat{\Theta} & I_m \end{pmatrix}$$

and again the result is immediate.

For $k = 2m + 1$ we take $A_2 = 0$, $A_1 = -I_m$ in (3.6); hence

$$F_{MN}J + \tfrac{1}{2}I = \begin{pmatrix} 0 & 0 \\ (I_{m+1} + \Phi^{-1}C)^{-1}\Phi^{-1}\Phi & (I_{m+1} + \Phi^{-1}C)^{-1} \end{pmatrix}$$

(note that $\tilde{A}_1 = I_{m+1}$). Hence

$$F_{MN} = \frac{1}{2}\begin{pmatrix} -I_m & 0 \\ 2(I_{m+1} + \Phi^{-1}C)^{-1}\Phi^{-1}\Theta & 2(I_{m+1} + \Phi^{-1}C)^{-1} - I_{m+1} \end{pmatrix} J^{-1},$$

$$= \frac{1}{2}\begin{pmatrix} -I_m & 0 \\ 2(I_{m+1} + \Phi^{-1}C)^{-1}\Phi^{-1}\Theta & (I_{m+1} - \Phi^{-1}C)(I_{m+1} + \Phi^{-1}C)^{-1} \end{pmatrix} J^{-1}. \tag{3.7}$$

Consider now the matrix $D = (I_{m+1} - \Phi^{-1}C)(I_{m+1} + \Phi^{-1}C)^{-1}$ which is uniformly bounded on $[a + \varepsilon, b^*)$ by Theorem B. The above equation for F_{MN} yields, after the indicated multiplication,

$$(F_{MN})_{m+1,m+1} = \tfrac{1}{2}D_{11}(-i).$$

Hence by (3.2),

$$2\,\mathrm{Im}(F_{MN})_{m+1,m+1} = -\mathrm{Re}\, D_{11} \leqslant 0;$$

thus $\mathrm{Re}\, D_{11} \geqslant 0$. The matrices $I_{m+1} \pm \Phi^{-1}C$ are lower triangular with all diagonal entries except the first equal to 1. Hence D is a lower triangular matrix with $D_{ii} = 1$ for $i > 1$ and $\mathrm{Re}\, D_{11} \geqslant 0$. From

$$D(I_{m+1} + \Phi^{-1}C) = I_{m+1} - \Phi^{-1}C$$

we obtain $\Phi^{-1}C = (I_{m+1} + D)^{-1}(I_{m+1} - D)$. Since $I_{m+1} + D$ is uniformly bounded, lower triangular, and $\mathrm{Re}(I_{m+1} + D)_{ii} \geqslant 1$ for all i, we have $(I_{m+1} + D)^{-1}$ uniformly bounded. Thus $\Phi^{-1}C$ is uniformly bounded and

$$\Phi^{-1}\Theta = (I_{m+1} + \Phi^{-1}C)[(I_{m+1} + \Phi^{-1}C)^{-1}\Phi^{-1}\Theta]$$

is uniformly bounded since it is the product of two uniformly bounded matrix functions; the proof is now complete for $k = 2m$ or $\mathrm{Im}\,\lambda > 0$; the case $\mathrm{Im}\,\lambda < 0$ and k odd follows from (3.11) below.

A differentiation shows that

$$Y(x, \lambda)^* JY(x, \bar{\lambda}) \equiv J,$$

from which it follows by reversing the order of the products that

$$J^{-1} = Y(x, \bar{\lambda})\, J^{-1}Y(x, \lambda)^*. \tag{3.8}$$

In the case $k = 2m$,

$$\begin{pmatrix} 0 & J_1^{-1} \\ -J_1^{*-1} & 0 \end{pmatrix} = Y(x, \bar{\lambda}) \begin{pmatrix} 0 & J_1^{-1} \\ -J_1^{*-1} & 0 \end{pmatrix} Y(x, \lambda)^*; \tag{3.9}$$

the upper left-hand corner of (3.9) is

$$0 = \Theta(x, \bar{\lambda}) J_1^{-1} \Phi^*(x, \lambda) - \Phi(x, \bar{\lambda}) J_1^{*-1} \Theta^*(x, \lambda)$$

from which we conclude that

$$J_1^* \Phi^{-1}(x, \bar{\lambda})\, \Theta(x, \bar{\lambda}) = [\Phi^{-1}(x, \lambda)\, \Theta(x, \lambda)]^* J_1. \tag{3.10}$$

In the case $k = 2m + 1$, similar calculations with (3.8) yield that

$$J_1^* \Phi^{-1}(x, \bar{\lambda})\, \Theta(x, \bar{\lambda}) = [\Phi^{-1}(x, \lambda)\, \Theta(x, \lambda)]^* \begin{bmatrix} 0 & i \\ J_1 & 0 \end{bmatrix}. \tag{3.11}$$

THEOREM 3.1. *If* (2.11) *holds, then the following hold for* $\operatorname{Im} \lambda \neq 0$.

(i) $M_\infty(\lambda) = -\lim_{x \to b^*} \Phi(x)^{-1}\, \Theta(x)$ *exists.*

(ii) $M_\infty(\lambda)$ *has rank* m, $M_\infty(\lambda)$ *is analytic on* $\operatorname{Im} \lambda \neq 0$, *and* $J_1^* M_\infty(\bar{\lambda}) = M_\infty(\lambda)^* J_1$ *if* $k = 2m$; $J_1^* M_\infty(\bar{\lambda}) = M_\infty(\lambda)^* \left[\begin{smallmatrix} 0 & i \\ J_1 & 0 \end{smallmatrix}\right]$ *if* $k = 2m + 1$ *and* $\operatorname{Im} \lambda > 0$.

(iii) *If* $\left[\begin{smallmatrix} X_\infty \\ \hat{X}_\infty \end{smallmatrix}\right] = \left[\begin{smallmatrix} \Theta & \Phi \\ \hat{\Theta} & \hat{\Phi} \end{smallmatrix}\right] \left[\begin{smallmatrix} I \\ M_\infty(\lambda) \end{smallmatrix}\right]$, *then the columns of* $\left[\begin{smallmatrix} X_\infty \\ \hat{X}_\infty \end{smallmatrix}\right]$ *form a basis of* $S(\lambda)$ *and* $\lim_{x \to b^*} \Phi^{-1}(x)\, X_\infty(x) = 0$.

(iv) *If* $k = 2m$, *then* $M_\infty(\lambda) = -\lim_{x \to b} \hat{\Phi}(x)^{-1}\, \hat{\Theta}(x)$.

Proof. From (3.10) and (3.11) it suffices to consider the case $\operatorname{Im} \lambda > 0$. By Lemma 3.1, to show the limit in (i) exists, it suffices to show all sequential limits are the same. Let $\{x_n\}$, $\{s_n\}$ be such that $x_n \to b^*$ as $n \to \infty$, $s_n \to b^*$ as $n \to \infty$, and

$$M_1 = -\lim_{n \to \infty} \Phi^{-1}(x_n)\, \Theta(x_n); \qquad M_2 = -\lim_{n \to \infty} \Phi^{-1}(s_n)\, \Theta(s_n).$$

Then by Lemmas 2.3 and 2.4, the columns of

$$\begin{pmatrix} X_1 \\ \hat{X}_1 \end{pmatrix} = Y \begin{pmatrix} I \\ M_1 \end{pmatrix}, \qquad \begin{pmatrix} X_2 \\ \hat{X}_2 \end{pmatrix} = Y \begin{pmatrix} I \\ M_2 \end{pmatrix}$$

are in $S(\lambda)$. Suppose there is a vector v so that $X_1(x)v \equiv 0$, $\hat{X}_1(x)v \equiv 0$. Using $x = a$ in the definition of $X_1, \hat{X}_1$ yields

$$0 = \begin{pmatrix} X_1(a) \\ \hat{X}_1(a) \end{pmatrix} v = Y(a) \begin{pmatrix} I \\ M_1 \end{pmatrix} v = \begin{pmatrix} v \\ M_1 v \end{pmatrix}$$

which implies $v = 0$; hence the columns of $\left[\begin{smallmatrix} X_1 \\ \hat{X}_1 \end{smallmatrix}\right]$ are linearly independent. Since there are the same number of columns as the dimension of $S(\lambda)$, they form a basis of $S(\lambda)$; similar remarks apply to $X_2, \hat{X}_2$. Thus there is a matrix K of rank equal to dim $S(\lambda)$ such that

$$\begin{pmatrix} X_1(x) \\ \hat{X}_1(x) \end{pmatrix} = \begin{pmatrix} X_2(x) \\ \hat{X}_2(a) \end{pmatrix} K.$$

Thus for $x = a$,

$$\begin{pmatrix} I \\ M_1 \end{pmatrix} = \begin{pmatrix} I \\ M_2 \end{pmatrix} K;$$

hence $K = I$ and $M_1 = M_2$. Since $\Phi^{-1}X_\infty = \Phi^{-1}\Theta + M_\infty(\lambda)$, it is immediate that $\Phi^{-1}X_\infty \to 0$ as $x \to b^*$.

The analyticity of M_∞ follows from the uniform boundedness of $\Phi^{-1}(b, \lambda)\, \Theta(b, \lambda)$ on compact λ-sets, and its convergence as $b \to b^*$.

If rank $M_\infty(\lambda) < m$, then there is a vector $v \neq 0$ such that $M_\infty(\lambda)v = 0$. In the case $k = 2m + 1$ and Im $\lambda < 0$, $M_\infty(\lambda)$ is an $m \times (m + 1)$ matrix (a linear transformation from $m + 1$ space into m space) and it is sufficient to suppose also $v_{m+1} = 0$. Lemma 2.4 then gives

$$0 < \int_a^{b^*} v^* \begin{bmatrix} X \\ \hat{X} \end{bmatrix}^* A \begin{bmatrix} X \\ \hat{X} \end{bmatrix} v \leqslant \frac{1}{2i \operatorname{Im} \lambda} \begin{bmatrix} v \\ M_\infty(\lambda)v \end{bmatrix}^* J \begin{bmatrix} v \\ M_\infty(\lambda)v \end{bmatrix}. \tag{3.12}$$

Calculation of the right-hand side of (3 .12) shows it to be zero in all cases. This contradiction establishes that rank $M_\infty(\lambda) = m$. The other properties of M in (ii) are immediate from (3.10) and (3.11).

From Lemma 3.1 we have that $\hat{\Phi}^{-1}\hat{\Theta}$ is uniformly bounded on $[a, b^*)$. Let $\{x_n\}$ be a sequence such that $x_n \to b^*$ as $n \to \infty$ and $-\hat{\Phi}^{-1}(x_n)\, \hat{\Theta}(x_n) \to M_0$ as $n \to \infty$. Define

$$\begin{pmatrix} Z_n \\ \hat{Z}_n \end{pmatrix} = \begin{pmatrix} \Theta & \Phi \\ \hat{\Theta} & \hat{\Phi} \end{pmatrix} \begin{pmatrix} I \\ -\hat{\Phi}^{-1}(x_n)\, \hat{\Theta}(x_n) \end{pmatrix}$$

Then $\hat{Z}_n(x_n) = 0$ and by Lemma 2.4, we have that the columns of

$$\begin{pmatrix} Z \\ \hat{Z} \end{pmatrix} = \begin{pmatrix} \Theta & \Phi \\ \hat{\Theta} & \hat{\Phi} \end{pmatrix} \begin{pmatrix} I \\ M_0 \end{pmatrix}$$

are in $S(\lambda)$; hence for some $m \times m$ matrix K

$$\begin{pmatrix} Z \\ \hat{Z} \end{pmatrix} = \begin{pmatrix} \Theta & \Phi \\ \hat{\Theta} & \hat{\Phi} \end{pmatrix} \begin{pmatrix} I \\ M_0 \end{pmatrix} = \begin{pmatrix} X_\infty \\ \hat{X}_\infty \end{pmatrix} K = \begin{pmatrix} \Theta & \Phi \\ \hat{\Theta} & \hat{\Phi} \end{pmatrix} \begin{pmatrix} I \\ M_\infty(\lambda) \end{pmatrix} K$$

from which it follows that $K=I$, $M_0=M_\infty(\lambda)$, and the proof of Theorem 3.1 is complete.

We now show also that $\Phi^{-1}C\to 0$ as $x\to b^*$ in (3.7). Let C_1 be a sequential limit of $\Phi^{-1}C$ and define

$$\begin{pmatrix} X_n \\ \hat{X}_n \end{pmatrix} = Y\begin{pmatrix} I_m \\ -(I_{m+1}-\Phi^{-1}(x_n)C)\,\Phi^{-1}(x_n)\,\Theta(x_n) \end{pmatrix},$$

where $\Phi^{-1}(x_n)C\to C_1$ as $n\to\infty$. Then $X_n(x_n)=+C\Phi^{-1}(x_n)\,\Theta(x_n)$ has its first m rows zero and is uniformly bounded; hence by the proof of Lemma 2.4,

$$\begin{pmatrix} X_1 \\ \hat{X}_1 \end{pmatrix} = Y\begin{pmatrix} I_m \\ (I_{m+1}-C_1)\,M_\infty(\lambda) \end{pmatrix}$$

has its columns in $S(\lambda)$. Then for some $m\times m$ matrix K,

$$\begin{pmatrix} X_1 \\ \hat{X}_1 \end{pmatrix} = \begin{pmatrix} X_\infty \\ \hat{X}_\infty \end{pmatrix} K$$

from which it follows as before that $K=I$ and $C_1=0$.

From the above and Theorem 3.1, we have that in cases (I), (II), and (III),

$$\lim_{b\to b^*} F_{MN}(\lambda, b) = F_M(\lambda, \infty)$$

exists. From (3.4), (3.5), and (3.6), we see that in both cases (I) and (II),

$$F_M(\lambda,\infty)J+\tfrac{1}{2}I$$
$$= \begin{pmatrix} A_2^*[A_2^*+M_\infty(\lambda)^{-1}A_1^*]^{-1} & A_2^*[-M_\infty(\lambda)A_2^*-A_1^*]^{-1} \\ -A_1^*[A_2^*+M_\infty(\lambda)^{-1}A_1^*]^{-1} & -A_1^*[-M_\infty(\lambda)A_2^*-A_1^*]^{-1} \end{pmatrix}. \tag{3.13}$$

In case (III),

$$F_M(\lambda,\infty)J+\tfrac{1}{2}I$$
$$= \begin{pmatrix} \tilde{A}_2[-M_\infty(\lambda)\tilde{A}_2+\tilde{A}_1]^{-1}M_\infty(\lambda) & \tilde{A}_2[-M_\infty(\lambda)\tilde{A}_2+\tilde{A}_1]^{-1} \\ \tilde{A}_1[-M_\infty(\lambda)\tilde{A}_2+\tilde{A}_1]^{-1}M_\infty(\lambda) & \tilde{A}_1[-M_\infty(\lambda)\tilde{A}_2+\tilde{A}_1]^{-1} \end{pmatrix}. \tag{3.14}$$

Clearly a central role is played in the above development by the uniform boundedness of the characteristic function F_{MN}. As applied to scalar equations, the results of Theorem 3.1 above agree with those in the work of Everitt [4–6]. In particular, the representation of the square-integrable solutions as certain linear combinations of nonsquare-integrable solutions is the same. The matrix formulation is not used in Everitt's work and he does not explicitly define the elements of $M_\infty(\lambda)$ as a limit in the manner described in this paper. However, it can be deduced from his theory. The theorey developed by Everitt does not use the uniform boundedness theorem of Atkinson but relies on the eigenvalues of certain Gram matrices to achieve existence and analyticity of the elements of $M_\infty(\lambda)$. An additional important aspect of Everitt's work on scalar equations is to relate the number of square-integrable solutions to the dimension of a limiting hypersurface (analogous to Weyl's circles).

We discuss now the behavior of $\Phi^{-1}\Theta$ when the equation may fail to be limit point. We state here results from the paper of Halvorsen [11] concerning the equation

$$x''(t) + (\lambda r(t) - s(t))\, x(t) = 0, \qquad a \leqslant t < \infty. \tag{3.15}$$

The nonreal solutions of (3.15) are written in polar form $x(t) = p(t)\, e^{i\omega(t)}$, and it is proved in [11] that $\omega'(t)$ is eventually of constant sign; hence $\lim \omega(t)$ as $t \to \infty$ exists in the extended sense. When this limit is finite the solution x is said to be of bounded argument. One result of [11] is that if for some λ, $\operatorname{Im} \lambda \neq 0$, (3.15) has a nontrivial solution of bounded argument, then (i) all solutions for all nonreal λ have bounded argument and (ii) for real λ, (3.15) is nonoscillatory, i.e., solutions have only finitely many zeros. This theorem yields a four-way classification of (3.15) at the singular point infinity: limit point or limit circle; bounded argument or unbounded argument.

In the limit-circle bounded argument case, it follows from [11, p. 17] that $\Phi^{-1}(t)\, \Theta(t)$ has a limit as $t \to \infty$. Since a limit-circle equation which is nonoscillatory for some real λ has solutions of bounded argument for nonreal λ [11, Theorem 4.2], examples are easy to construct, e.g.,

$$x''(t) + \lambda t^{-4} x(t) = 0, \qquad 1 \leqslant t < \infty.$$

In the limit-circle unbounded argument case, it follows from [11, p. 18] that

$$0 < \liminf_{t \to \infty} |\Phi^{-1}(t)\, \Theta(t)| < \limsup_{t \to \infty} |\Phi^{-1}(t)\, \Theta(t)|.$$

An example of this behavior is the equation

$$x''(t) + (\lambda + e^{2t})\, x(t) = 0$$

which has the Bessel function solutions $J_{is}(e^t)$ and $J_{-is}(e^t)$, where $\lambda = s^2$, $s = u + iv$. We have

$$\text{argument}\, J_{is}(e^t) = e^t - \pi/4 + \pi v/2 + o(1),$$

$$\text{argument}\, J_{-is}(e^t) = -e^t - \pi/4 + \pi v/2 + o(1),$$

Further use of the asymptotic modulus of $J_{is}(e^t)$ and $J_{-is}(e^t)$ shows that the sequential limits of $\Phi^{-1}(t)\,\Theta(t)$ form a circle.

In the limit-point unbounded argument case $\Phi^{-1}(t)\,\Theta(t)$ will also spiral. An example of this is the Airy equation

$$x''(t) + (\lambda - t)\,x(t) = 0$$

whose solutions are the Airy function $\mathrm{Ai}(t-\lambda)$ and $\mathrm{Bi}(t-\lambda)$. The function $\mathrm{Ai}(t-\lambda)$ has the asymptotic form

$$\mathrm{Ai}(t-\lambda) = K(t-\lambda)^{-1/4} \exp[-(\tfrac{2}{3})(t-\lambda)^{3/2}]$$

and a direct argument for $\operatorname{Im}\lambda \neq 0$ shows that the argument of $\mathrm{Ai}(t-\lambda)$ is unbounded.

We now show that in the case of the Hamiltonian system (1.2), the solution $\binom{X_\infty}{\hat{X}_\infty}$ of Theorem 3.1(iii) may be properly defined as the principal solution. If $\binom{W}{\hat{W}}$ is a solution of (1.2) such that

$$\begin{pmatrix} W & X_\infty \\ \hat{W} & \hat{X}_\infty \end{pmatrix} = \begin{pmatrix} \Theta & \Phi \\ \hat{\Theta} & \hat{\Phi} \end{pmatrix} \begin{pmatrix} C_1 & I \\ C_2 & M_\infty(\lambda) \end{pmatrix}$$

is a fundamental matrix, then $-M_\infty(\lambda)\,C_1 + C_2$ is nonsingular. This follows from the fact that $-M_\infty(\lambda)\,C_1 v + C_2 v = 0$ implies that

$$\begin{pmatrix} C_1 & I \\ C_2 & M_\infty(\lambda) \end{pmatrix} \begin{pmatrix} v \\ -C_1 v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

which implies $v = 0$. Hence for

$$\begin{pmatrix} Z \\ \hat{Z} \end{pmatrix} = \begin{pmatrix} \Theta & \Phi \\ \hat{\Theta} & \hat{\Phi} \end{pmatrix} \begin{pmatrix} D_1 \\ D_2 \end{pmatrix},$$

$$\lim_{x \to b^*} W^{-1}(x)\,Z(x) = \lim_{x \to b^*} [\Phi^{-1}(x)\,\Theta(x)\,C_1 + C_2]^{-1}\,[\Phi^{-1}(x)\,\Theta(x)\,D_1 + D_2]$$

$$= [-M_\infty(\lambda)\,C_1 + C_2]^{-1}\,[-M_\infty(\lambda)\,D_1 + D_2]. \qquad (3.16)$$

Similarly, $\lim_{x \to b^*} \hat{W}^{-1}(x)\,\hat{Z}(x) = [-M_\infty(\lambda)\,C_1 + C_2]^{-1}\,[-M_\infty(\lambda)\,D_1 + D_2]$. For $Z = X_\infty$, $\hat{Z} = \hat{X}_\infty$, the limit in (3.16) is zero.

Conversely, suppose

$$\begin{bmatrix} W & Z \\ \hat{W} & \hat{Z} \end{bmatrix} = \begin{bmatrix} \Theta & \Phi \\ \hat{\Theta} & \hat{\Phi} \end{bmatrix} \begin{bmatrix} C_1 & D_1 \\ C_2 & D_2 \end{bmatrix}, \qquad \det \begin{bmatrix} C_1 & D_1 \\ C_2 & D_2 \end{bmatrix} \neq 0,$$

W is invertible for x sufficiently near b^*, and $W^{-1}(x)\,Z(x) \to 0$ as $x \to b^*$. Since from (3.16)

$$W^{-1}(x)\,Z(x) = [\Phi^{-1}(x)\,\Theta(x)\,C_1 + C_2]^{-1}\,[\Phi^{-1}(x)\,\Theta(x)\,D_1 + D_2],$$

we have that

$$\begin{aligned}\lim_{x \to b^*} [\Phi^{-1}(x)\,\Theta(x)\,C_1 + C_2]\,W^{-1}(x)\,Z(x) &= [-M_\infty(\lambda)\,C_1 + C_2]0 = 0 \\ &= -M_\infty(\lambda)\,D_1 + D_2;\end{aligned}$$

hence $D_2 = M_\infty(\lambda)\,D_1$ and $Z(x) = X_\infty(x)\,D_1$. Thus we have a characterization of X_∞ as the smallest solution which parallels the characterization of the principal solution for real λ [2, p. 43; 12, p. 355; 15, Chap. 7, §5].

An examination of the properties of the principal solution for $\lambda = 0$ and the Hamiltonian system (1.2) being nonoscillatory shows that the argument depends on use of a reduction formula [cf. 2, p. 35] and the monotonicity of a certain integral. The reduction formula fails to hold for nonreal λ. However, if in (1.2), the matrices A, B, C and K are real with $B^T = B$, $C^T = C$, and $K = K^T$, the reduction formula holds with $*$ replaced by transpose. Using this formula, one obtains that for $\operatorname{Im}\lambda \neq 0$,

$$\Phi(x) = X_\infty(x)\,S_0(x)N,$$

where

$$S_0(x) = \int_a^x X_\infty^{-1}(s)\,B(s)\,X_\infty^{T-1}(s)\,ds,$$

$$N \equiv X_\infty^T(x)\,\hat{\Phi}(x) - \hat{X}_\infty^T(x)\,\Phi(x) = X_\infty^T(a) = I.$$

Thus we have that $S_0^{-1}(x) \to 0$ as $x \to b^*$, although $S_0(x)$ is not a monotone matrix function as in the usual theory of principal solutions of Hamiltonian systems.

4. Inhomogeneous Problems

In this section we study the well-posedness of the problem

$$Jy' = [\lambda A(x) + B(x)]\,y - f, \qquad a \leqslant x < b^*, \tag{4.1}$$

together with boundary conditions, where f belongs to the class $\mathscr{F}$ of locally integrable $k \times 1$ vector functions on $[a, b^*)$ which also satisfy

$$\int_a^{b^*} \left\| \begin{bmatrix} X_\infty(t,\lambda) \\ \hat{X}_\infty(t,\lambda) \end{bmatrix}^* f(t) \right\| dt < \infty, \qquad \operatorname{Im}\lambda \neq 0. \tag{4.2}$$

Here $X_\infty(t,\lambda)=\Theta(t,\lambda)+\Phi(t,\lambda)M_\infty(\lambda)$, and $M_\infty(\lambda)=\lim_{b\to b'}M_b(\lambda)$ is the M-function arising from Theorem 3.1.

We continue to consider only cases (I), (II), (III). We make the further simplification in this section that $A_2=0$ and $A_1=I$, there being no real difficulty in extending our results to the more general setting. Following Atkinson we define the kernel $K(x,t,\lambda)$ by

$$\begin{aligned} K(x,t,\lambda)&=Y(x,\lambda)\{F_M(\lambda,\infty)-(\tfrac{1}{2})J^{-1}\}\,Y(t,\bar\lambda)^*, \qquad x>t,\\ K(x,t,\lambda)&=Y(x,\lambda)\{F_M(\lambda,\infty)+(\tfrac{1}{2})J^{-1}\}\,Y(t,\bar\lambda)^*, \qquad x<t \end{aligned}\tag{4.3}$$

for $\operatorname{Im}(\lambda)\neq 0$. By (3.13) and (3.14), in the cases we consider, we have

$$F_M(\lambda,\infty)=\frac{1}{2}\begin{bmatrix}-I & 0\\ -2M_\infty(\lambda) & I\end{bmatrix}J^{-1}.$$

Hence $K(x,t,\lambda)$ will have either of two forms, according as k is even or odd. If k is even we find

$$\begin{aligned} F_M(\lambda,\infty)+\frac{1}{2}J^{-1}&=\frac{1}{2}\left\{\begin{bmatrix}-I & 0\\ -2M_\infty(\lambda) & I\end{bmatrix}+\begin{bmatrix}I & 0\\ 0 & I\end{bmatrix}\right\}J^{-1}\\ &=\begin{bmatrix}0 & 0\\ -M_\infty(\lambda) & I\end{bmatrix}\begin{bmatrix}0 & J_1^{-1}\\ -J_1^{*-1} & 0\end{bmatrix}\\ &=\begin{bmatrix}0 & 0\\ -J_1^{*-1} & -M_\infty(\lambda)J_1^{-1}\end{bmatrix}, \end{aligned}$$

while

$$\begin{aligned} F_M(\lambda,\infty)-(\tfrac{1}{2})J^{-1}&=\begin{bmatrix}-I & 0\\ -M_\infty(\lambda) & 0\end{bmatrix}\begin{bmatrix}0 & J_1^{-1}\\ -J_1^{*-1} & 0\end{bmatrix}\\ &=\begin{bmatrix}0 & -J_1^{-1}\\ 0 & -M_\infty(\lambda)J_1^{-1}\end{bmatrix}. \end{aligned}$$

For $x<t$, (4.3) becomes

$$\begin{aligned} K(x,t,\lambda)&=\begin{bmatrix}\Theta(x,\lambda) & \Phi(x,\lambda)\\ \hat\Theta(x,\lambda) & \hat\Phi(x,\lambda)\end{bmatrix}\begin{bmatrix}0 & 0\\ -J_1^{*-1} & -M_\infty(\lambda)J_1^{-1}\end{bmatrix}\\ &\quad\times\begin{bmatrix}\Theta(t,\bar\lambda)^* & \hat\Theta(t,\bar\lambda)^*\\ \Phi(t,\bar\lambda)^* & \hat\Phi(t,\bar\lambda)^*\end{bmatrix}\\ &=\begin{bmatrix}-\Phi(x,\lambda)J_1^{*-1}X_\infty(t,\bar\lambda)^* & -\Phi(x,\lambda)J_1^{*-1}\hat X_\infty(t,\bar\lambda)^*\\ -\hat\Phi(x,\lambda)J_1^{*-1}X_\infty(t,\bar\lambda)^* & -\hat\Phi(x,\lambda)J_1^{*-1}\hat X_\infty(t,\bar\lambda)^*\end{bmatrix}\\ &=-\begin{bmatrix}\Phi(x,\lambda)J_1^{*-1}\\ \hat\Phi(x,\lambda)J_1^{*-1}\end{bmatrix}\begin{bmatrix}X_\infty(t,\bar\lambda)\\ \hat X_\infty(t,\bar\lambda)\end{bmatrix}^*, \end{aligned}$$

where we have used $M_\infty(\lambda) J_1^{-1} = J_1^{*-1} M_\infty(\bar{\lambda})^*$ from Theorem 3.1. If $x > t$, an analogous calculation gives

$$K(x, t, \lambda) = -\begin{bmatrix} X_\infty(x, \lambda) \\ \hat{X}_\infty(x, \lambda) \end{bmatrix} \begin{bmatrix} \Phi(t, \bar{\lambda}) J_1^{*-1} \\ \hat{\Phi}(t, \bar{\lambda}) J_1^{*-1} \end{bmatrix}^*,$$

and we note the symmetry relation $K(x, t, \lambda) = K(t, x, \bar{\lambda})^*$.

If k is odd, that is, if case (III) holds, one calculates

$$K(x, t, \lambda) = -\begin{bmatrix} \Phi(x, \lambda) \hat{J}_1^{*-1} \\ \hat{\Phi}(x, \lambda) \hat{J}_1^{*-1} \end{bmatrix} \begin{bmatrix} X_\infty(t, \bar{\lambda}) \\ \hat{X}_\infty(t, \bar{\lambda}) \end{bmatrix}^*, \qquad x < t,$$

and

$$K(x, t, \lambda) = -\begin{bmatrix} X_\infty(x, \lambda) \\ \hat{X}_\infty(x, \lambda) \end{bmatrix} \begin{bmatrix} \Phi(t, \bar{\lambda}) J_1^{*-1} \\ \hat{\Phi}(t, \bar{\lambda}) J_1^{*-1} \end{bmatrix}^*, \qquad x > t,$$

where

$$\hat{J}_1 = \begin{bmatrix} 0 & i \\ J_1 & 0 \end{bmatrix}.$$

Now let $f \in \mathscr{F}$ and define $y(x) = y(x, \lambda)$ by

$$y(x) = \int_a^{b^*} K(x, t, \lambda) f(t)\, dt. \tag{4.4}$$

By differentiation,

$$Jy' - [\lambda A(x) + B(x)]\, y = -J\{K(x, x+0, \lambda) - K(x, x-0, \lambda)\}\, f(x), \tag{4.5}$$

where $x + 0$ and $x - 0$ denote one-sided limits. Differentiation here is understood to be in the Lebesgue sense. It follows from (4.3) that $K(x, t, \lambda)$ has a jump discontinuity in the t variable across $t = x$ of value $Y(x, \lambda) J^{-1} Y(x, \bar{\lambda})^*$. However, this equals J^{-1} by (3.8), and so the right side of (4.5) reduces to $-f(x)$. This shows that $y(x)$ given by (4.4) is a solution to the differential equation (4.1).

In what follows, a solution $y(x)$ of (4.1) will be row-partitioned according to (2.16)–(2.18); i.e.,

$$y(x) = \begin{bmatrix} y_0(x) \\ \hat{y}_0(x) \end{bmatrix},$$

where $y_0(x)$ has the same number of rows as Θ, Φ, X.

THEOREM 4.1. *If $f \in \mathscr{F}$, then the boundary value problem*

$$Jy' = [\lambda A(x) + B(x)]\, y - f(x), \qquad y_0(a) = v_0,$$

$$\int_a^{b^*} y^*(x)\, A(x)\, y(x)\, dx < \infty, \tag{4.6}$$

has at most one solution.

Proof. It is enough to consider only the homogeneous problem $f \equiv 0$, and with boundary condition $y_0(a) = 0$. Since the columns of $[X_\infty^*(x, \lambda), \hat{X}_\infty^*(x, \lambda)]^*$ form a basis of $S(\lambda)$, we must have

$$y(x) = \begin{bmatrix} X_\infty(x, \lambda) \\ \hat{X}_\infty(x, \lambda) \end{bmatrix} C,$$

where C is a constant column vector. Thus

$$0 = y_0(a) = X_\infty(a, \lambda) C = IC = C,$$

which completes the proof.

To prove existence of solutions of the problem (4.6) in the general setting of Hamiltonian systems, we require further hypotheses. We shall first assume, in addition to (2.11), that

$$\lim_{x \to b^*} y^*(x)\, Jy(x) = 0 \tag{4.7}$$

for all solutions y of (2.1) which are of integrable square; i.e., for which (2.8) holds. In the case where (2.1) represents a scalar differential equation, (4.7) is implied by (2.11) [14]. Additionally f will be replaced in (4.6) by $A(x)\, g(x)$, where g is of integrable square. In view of the scalar case, this is the natural boundary problem to consider (see [18]). Note that for such g, $f = Ag \in \mathscr{F}$ is a consequence of the Cauchy–Schwarz inequality.

THEOREM 4.2. *Suppose that* (2.11) *and* (4.7) *hold, and that g satisfies* (2.8). *Then the boundary value problem*

$$Jy' = [\lambda A(x) + B(x)]\, y - A(x)\, g(x), \qquad y_0(a) = v_0,$$

$$\int_a^{b^*} y^*(x)\, A(x)\, y(x)\, dx < \infty \tag{4.8}$$

has a unique solution.

Proof. Uniqueness was proved in Theorem 4.1. Our existence proof parallels that for scalar equations in [13, p. 558–559].

Define

$$y(x) = \begin{bmatrix} X_\infty(t, \lambda) \\ \hat{X}_\infty(t, \lambda) \end{bmatrix} v_0 + \int_a^{b^*} K(x, t, \lambda) A(t) g(t) \, dt, \qquad a \leqslant x < b^*. \quad (4.9)$$

We have shown that this is a solution to (4.8), so there remains only to prove that $y(x)$ satisfies (2.8). It is sufficient that $v_0 = 0$.

Let $\{t_n\}_1^\infty$ be a monotone increasing sequence in $[a\, b^*)$ with $t_n \to b^*$ as $n \to \infty$. Define the sequences $\{g_n(t)\}$ and $\{y_n(x)\}$ by

$$\begin{aligned} g_n(t) &= g(t), \qquad && a \leqslant t \leqslant t_n, \\ &= 0, && t > t_n \end{aligned}$$

and

$$y_n(x) = \int_a^{b^*} K(x, t, \lambda) A(t) \, g_n(t) \, dt, \qquad n = 1, 2, 3, \ldots.$$

We have

$$y_n(x) - y(x) = -\int_{t_n}^{b^*} K(x, t, \lambda) A(t) \, g(t) \, dt.$$

If k is even and $x \leqslant t_n$, then our previous calculation of $K(x, t, \lambda)$ reveals

$$y_n(x) - y(x) = \begin{bmatrix} \Phi(x, \lambda) J_1^{*-1} \\ \hat{\Phi}(x, \lambda) J_1^{*-1} \end{bmatrix} \int_{t_n}^{b^*} \begin{bmatrix} X_\infty(t, \bar{\lambda}) \\ \hat{X}_\infty(t, \bar{\lambda}) \end{bmatrix}^* A(t) \, g(t) \, dt.$$

If k is odd and $x \leqslant t_n$, a similar formula holds with J_1 replaced by $\hat{J}_1$. As a result, $y_n(x) \to y(x)$ uniformly on compact subsets of $[a, b^*)$.

Observe now that, for $x \geqslant t_n$,

$$\begin{aligned} y_n(x) &= \int_a^{t_n} K(x, t, \lambda) A(t) \, g(t) \, dt \\ &= -\begin{bmatrix} X_\infty(x, \lambda) \\ \hat{X}_\infty(x, \lambda) \end{bmatrix} \int_a^{t_n} \begin{bmatrix} \Phi(t, \bar{\lambda}) J_1^{*-1} \\ \hat{\Phi}(t, \bar{\lambda}) J_1^{*-1} \end{bmatrix}^* A(t) \, g(t) \, dt, \end{aligned}$$

and so each $y_n(x)$ is of integrable square.

Multiplying (4.1) by $y^*(x)$ and integrating by parts gives

$$y^*Jy(b) - y^*Jy(a) + (\bar{\lambda} - \lambda) \int_a^b y^*Ay \, dx = \int_a^b (f^*y - y^*f) \, dx. \tag{4.10}$$

We put $y=y_n$ and $f=Ag_n$ in this equation. First note that

$$\begin{aligned}(y_n^*Jy_n)(a) &= iG_n^*W_2^*(a)\,W_2(a)\,G_n, \qquad k=2m+1, \quad \operatorname{Im}\lambda>0,\\ &=0 \qquad\qquad\qquad\qquad\qquad \text{otherwise,}\end{aligned} \tag{4.11}$$

where $G_n=\int_a^{b^*}[X_\infty^*, \hat{X}_\infty^*]^*\,Ag_n\,dt$ and W_2 is defined in Lemma 2.2, case (ii). Then (4.10) becomes

$$\int_a^b y_n^*Ay_n\,dx=\frac{1}{2i\operatorname{Im}\lambda}\left\{\int_a^b [y^*Ag_n-(y_n^*Ag_n)^*]+y_n^*Jy_n(b)-i\tau_n\right\},$$

where $i\tau_n=(y_n^*Jy_n)(a)$ is given by (4.11). Note that $\{\tau_n/\operatorname{Im}\lambda\}\geqslant 0$ in all cases, and therefore

$$\int_a^b y_n^*Ay_n\,dx\leqslant\frac{1}{2i\operatorname{Im}\lambda}\left\{\int_a^b [y_n^*Ag_n-(y_n^*Ag_n)^*]+y_n^*Jy_n(b)\right\}.$$

Letting $b\to b^*$ and using hypothesis (4.7) now yields

$$\int_a^{b^*} y_n^*Ay_n\,dx\leqslant\frac{1}{\operatorname{Im}\lambda}\operatorname{Im}\left\{\int_a^{b^*} y_n^*Ag_n\right\}$$

(recall that $y_n^*Ag_n$ is a complex number). By the Cauchy–Schwarz inequality, we have

$$\int_a^{b^*} y_n^*Ay_n\,dx\leqslant\left|\frac{1}{\operatorname{Im}\lambda}\right|\left(\int_a^{b^*} y_n^*Ay_n\,dx\right)^{1/2}\left(\int_a^{b^*} g_n^*Ag_n\,dx\right)^{1/2}$$

or

$$\int_a^{b^*} y_n^*Ay_n\,dx\leqslant\left|\frac{1}{\operatorname{Im}\lambda}\right|^2\int_a^{b^*} g_n^*Ag_n\,dx.$$

Since g is of integrable square, the terms $\int_a^{b^*} y_n^*Ay_n\,dx$ are bounded as $n\to\infty$. By Fatou's lemma, $\int_a^{b^*} y^*Ay\,dx<\infty$, and this completes the proof.

Define the operator T by

$$Ty=-Jy'+[\lambda A+B]\,y.$$

If y is such that $Ty=f\in\mathscr{F}$, then both $y(x)$ and the function $w(x)=\int_a^{b^*}K(x,t,\lambda)\,f(t)\,dt$ are solutions of (4.1) on $[a,b^*)$. This leads to the identity, valid when $Ty\in\mathscr{F}$,

$$y(x) = \begin{bmatrix} X_\infty(x, \lambda) \\ \hat{X}_\infty(x, \lambda) \end{bmatrix} c_1 + \begin{bmatrix} \Phi(x, \lambda) \\ \hat{\Phi}(x, \lambda) \end{bmatrix} c_2 + \int_a^{b^*} K(x, t, \lambda)(Ty)(t)\, dt, \tag{4.12}$$

where c_1 and c_2 are constant matrices.

Let $\mathscr{F}'$ be the subclass of $\mathscr{F}$ consisting of all $f \in \mathscr{F}$ for which

$$\int_a^{b^*} \left\| \begin{bmatrix} \Phi(t, \bar{\lambda})\, J_1^{*-1} \\ \hat{\Phi}(t, \bar{\lambda})\, J_1^{*-1} \end{bmatrix}^* f(t) \right\| dt < \infty.$$

Then we obtain the following theorem which is analogous to a result in [16, Corollary 2.5].

THEOREM 4.3. *Let y be such that $Ty \in \mathscr{F}'$. Then the component $y_0(x)$ is such that the limit*

$$\lim_{x \to b^*} \Phi^{-1}(x, \lambda)\, y_0(x)$$

exists. If, in addition, each of the functions $y(x)$ and $\int_a^{b^*} K(x, t, \lambda)(Ty)(t)\, dt$ is of integrable square, then the above limit has the (matrix) values 0.

Proof. Suppose first that k is even. Writing out the first component of (4.12), we have

$$y_0(x) = X_\infty(x, \lambda)\, c_1 + \Phi(x, \lambda)\, c_2 - X_\infty(x, \lambda) \int_a^x \begin{bmatrix} \Phi(t, \bar{\lambda})\, J_1^{*-1} \\ \hat{\Phi}(t, \bar{\lambda})\, J_1^{*-1} \end{bmatrix}^* (Ty)(t)\, dt$$

$$- \Phi(x, \lambda)\, J_1^{*-1} \int_x^{b^*} \begin{bmatrix} X_\infty(t, \bar{\lambda}) \\ \hat{X}_\infty(t, \bar{\lambda}) \end{bmatrix}^* (Ty)(t)\, dt.$$

A similar formula holds when k is odd. Now multiply by $\Phi^{-1}(x, \lambda)$, let $x \to b^*$ and use Theorem 3.1(iii) to obtain

$$\lim_{x \to b^*} \Phi^{-1}(x, \lambda)\, y_0(x) = c_2.$$

If $y(x)$ and $\int_a^{b^*} K(x, t, \lambda)(Ty(t)\, dt$ are of integrable square, then $c_2 \neq 0$ implies that a nontrivial linear combination of the columns of $[\Phi^*, \hat{\Phi}^*]^*$ is of integrable square. This is a contradiction since the matrix

$$\begin{bmatrix} X_\infty & \Phi \\ \hat{X}_\infty & \hat{\Phi} \end{bmatrix}$$

is a fundamental matrix. This completes the proof.

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